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On a Theorem for Expanding Functions of Functions.

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To expand any function of fx, say ϕfx, where

$$fx = a + bx + \frac{1}{2}cx^2 + \frac{1}{2.3}dx^3 + \dots,$$

it is customary to employ the method of derivation devised by Arbogast. By this method,

$$\phi fx = \phi a + xd\phi a + \frac{1}{2}x^2d^2\phi a + \dots,$$

where, after the differentiations indicated are performed, the differentials da, d^2a, d^3a, ..., are to be replaced by b, c, d, ... A more direct solution of the problem will, I think, be derived, as a special case, from the following general theorem, which would seem to be new:

$$\phi\,(fx, f'x, f''x \dots) = \left(1 + x\mathrm{D} + \frac{1}{2}x^2\mathrm{D}^2 + \frac{1}{2.3}x^3\mathrm{D}^3 + \dots\right)\phi\,(a, b, c \dots). \quad (1)$$

Here

$$\mathrm{D} = b\frac{d}{da} + c\frac{d}{db} + d\frac{d}{dc} + \dots,{}^*$$

D^n represents n successive performances of the operation D, and

$$f'x = \frac{dfx}{dx} = b + cx + \frac{1}{2}dx^2 + \frac{1}{2.3}ex^3 + \dots,$$

$$f''x = c + dx + \frac{1}{2}ex^2 + \frac{1}{2.3}fx^3 + \dots,$$

and so on. To prove this theorem, which is a simple consequence of that of Maclaurin, it is only necessary to write

$$\phi\,(fx, f'x, f''x \dots) = \left(1 + x\frac{d}{d0} + \frac{1}{2}x^2\frac{d^2}{d0^2} + \dots\right)\phi\,(f0, f'0, f''0 \dots),$$

and to observe that, as regards any function of $f0, f'0$ ..., say ψ,

$$\frac{d\psi}{d0} = f'0\frac{d\psi}{df0} + f''0\frac{d\psi}{df'0} + \dots,$$

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* The letter d is here used in two senses, neither of which is likely to be misunderstood. The same may be said of the letter f.

whence, since $f0$, $f'0$, &c., are respectively $a, b \ldots$, the symbol $\frac{d}{d0}$ may be replaced by D.

A more detailed demonstration may be derived from Taylor's theorem. Thus, if z represent $\phi\,(fz, f'z, f''z \ldots)$,

$$\phi\,[f(z+x), f'(z+x), \ldots] = \mathrm{z} + x\frac{d\mathrm{z}}{dz} + \frac{1}{2}x^2\frac{d^2\mathrm{z}}{dz^2} + \ldots . \tag{2}$$

The operation $\frac{d}{dz}$ is to be performed successively on z, $\frac{d\mathrm{z}}{dz}$, $\frac{d^2\mathrm{z}}{dz^2} \ldots$, all functions of $fz, f'z \ldots$, and may therefore be decomposed, on the principle of partial differentiation, into

$$\frac{d}{dz} = f'z\frac{d}{dfz} + f''z\frac{d}{df'z} + \ldots .$$

Let $z = 0$, and let $\frac{d}{dz}$ be represented by D; then, since $f0 = a$, $f'0 = b \ldots$,

$$\mathrm{D} = b\frac{d}{da} + c\frac{d}{db} + \ldots ,$$

and (2) becomes

$$\phi\,(fx, f'x \ldots) = (1 + x\mathrm{D} + \frac{1}{2}x^2\mathrm{D}^2 + \ldots)\,\phi\,(a, b \ldots).$$

The "remainder after n terms" is, of course, $\frac{1}{n!}x^n\mathrm{D}^n\phi\,(fx_1, f'x_1 \ldots)$, where $x_1 = \theta x$, θ being some proper fraction.

The most important special case of the general theorem is this,

$$\phi fx = \phi a + x\mathrm{D}\phi a + \frac{1}{2}x^2\mathrm{D}^2\phi a + \frac{1}{2.3}x^3\mathrm{D}^3\phi a + \ldots . \tag{3}$$

Here, of course, as before,

$$fx = a + bx + \frac{1}{2}cx^2 + \frac{1}{2.3}dx^3 + \ldots ,$$

$$\mathrm{D} = b\frac{d}{da} + c\frac{d}{db} + \ldots .$$

This formula will be found, I think, superior in directness to Arbogast's method, while yielding the same results for the same expenditure of labor.

If, in (3), we put $b = 1$, and $c = d = \ldots = 0$, we have remaining, as a special case, Taylor's theorem.

Both (1) and (3) may be modified as follows. For a, write α; for b, β; for c, 2γ; for d, 2.3δ; and so on. Then

$$
\begin{aligned}
fx &= \alpha + \beta x + \gamma x^2 + \delta x^3 + \ldots, \\
f'x &= \beta + 2\gamma x + 3\delta x^2 + 4\varepsilon x^3 + \ldots, \\
f''x &= 2\gamma + 2.3\delta x + 3.4\varepsilon x^2 + 4.5\zeta x^3 + \ldots, \\
f'''x &= 2.3\delta + 2.3.4\varepsilon x + 3.4.5\zeta x^2 + \ldots.
\end{aligned}
$$

and so on, and

$$\mathrm{D} = \beta\frac{d}{d\alpha} + 2\gamma\frac{d}{d\beta} + 3\delta\frac{d}{d\gamma} + 4\varepsilon\frac{d}{d\delta} + \ldots.$$

Let

$$\psi(fx, f_1x, f_2x \ldots) = \phi(fx, f'x, f''x \ldots),$$

where

$$
\begin{aligned}
fx &= \alpha + \beta x + \gamma x^2 + \delta x^3 + \ldots, \\
f_1x &= f'x = \beta + 2\gamma x + 3\delta x^2 + \ldots, \\
f_2x &= \frac{1}{2}f''x = \gamma + 3\delta x + 6\varepsilon x^2 + \ldots, \\
f_3x &= \frac{1}{2.3}f'''x = \delta + 4\varepsilon x + 10\zeta x^2 + \ldots,
\end{aligned}
$$

and so on. The law of the numerical coefficients is obvious. Each may be found by adding the one above to the one on the left hand. Then (1) assumes this form,

$$\psi(fx, f_1x, f_2x \ldots) = (1 + x\mathrm{D} + \frac{1}{2}x^2\mathrm{D}^2 + \ldots)\,\psi(\alpha, \beta, \gamma \ldots), \qquad (4)$$

and (3) becomes

$$\psi fx = \psi\alpha + x\mathrm{D}\psi\alpha + \frac{1}{2}x^2\mathrm{D}^2\psi\alpha + \frac{1}{2.3}x^3\mathrm{D}^3\psi\alpha + \ldots. \qquad (5)$$

In these theorems, as has just been stated,

$$\mathrm{D} = \beta\frac{d}{d\alpha} + 2\gamma\frac{d}{d\beta} + 3\delta\frac{d}{d\gamma} + \ldots.$$

Occasions may arise when these modifications may be found more convenient in use than the forms first given.

The formulæ thus far considered hold good, not only when fx is an algebraic function, but also in cases where fx is an infinite series. Let us now assume that fx contains no power of x greater than x^n; in other words, let

$$fx = a + bx + \ldots + \frac{1}{n-1!}jx^{n-1} + \frac{1}{n!}kx^n.$$

Having thus restricted the meaning of fx, let a be replaced by α; b by $n\beta$; c by $n^{2)}\gamma$; d by $n^{3)}\delta$; and so on, where $n^{r)}$ represents the factorial $n(n-1)(n-2)\ldots(n-r+1)$. Then

$$\begin{aligned} fx &= \alpha + n\beta x + \frac{1}{2}\, n^{2)}\gamma x^2 + \ldots + \varkappa x^n, \\ f'x &= n\beta + n^{2)}\gamma x + \frac{1}{2}\, n^{3)}\delta x^2 + \ldots + n\varkappa x^{n-1}, \\ f''x &= n^{2)}\gamma + n^{3)}\delta x + \ldots + n^{2)}\varkappa x^{n-2}, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f^{(n-1)}x &= n^{n-1)}\iota + n^{n)}\varkappa x, \\ f^{(n)}x &= n^{n)}\varkappa . \end{aligned}$$

Let

$$\psi\,(fx, f_1x, \ldots f_{n-1}x, f_nx) = \phi\,(fx, f'x, \ldots f^{(n-1)}x, f^{(n)}x),$$

where

$$\begin{aligned} fx &= \alpha + n\beta x + \frac{1}{2}\, n^{2)}\gamma x^2 + \ldots + \varkappa x^n, \\ f_1x &= \frac{1}{n}\, f'x = \beta + (n-1)\,\gamma x + \frac{1}{2}\,(n-1)^{2)}\delta x^2 + \ldots + \varkappa x^{n-1}, \\ f_2x &= \frac{1}{n^{2)}}\, f''x = \gamma + (n-2)\,\delta x + \frac{1}{2}\,(n-2)^{2)}\varepsilon x^2 + \ldots + \varkappa x^{n-2}, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_{n-1}x &= \frac{1}{n^{n-1)}}\, f^{(n-1)}x = \iota + \varkappa x, \\ f_nx &= \frac{1}{n^{n)}}\, f^{(n)}x + \varkappa. \end{aligned}$$

Then

$$\psi\,(\alpha, \beta, \gamma \ldots \iota, \varkappa) = \phi\,(a, b, c \ldots j, k)\,;$$

and, since $\frac{d}{da} = \frac{d}{d\alpha}$, $\frac{d}{db} = \frac{d}{dn\beta} = \frac{1}{n}\,\frac{d}{d\beta}$, and so on,

$$\mathsf{D} = n\beta\,\frac{d}{d\alpha} + (n-1)\,\gamma\,\frac{d}{d\beta} + (n-2)\,\delta\,\frac{d}{d\gamma} + \ldots + 2\iota\,\frac{d}{d\theta} + \varkappa\,\frac{d}{d\iota}.$$

Making these several substitutions in (1), we derive a known result, due to Dr. Salmon,

$$\psi\,(fx, f_1x, \ldots f_{n-1}x, f_nx) = (1 + x\mathsf{D} + \frac{1}{2}\,x^2\mathsf{D}^2 + \ldots)\,\psi\,(\alpha, \beta, \ldots \iota, \varkappa). \qquad (6)$$

This theorem, to which the name of Salmon's Theorem* has been assigned, is usually given in the opposite order, as, for example, in the following statement:† "If f is any function of $a, b, c \ldots$, and f_1 the same function of

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* This name was assigned by me, without consulting Dr. Salmon, in error. The theorem, as far as I have been able to ascertain, seems first to have been stated and proved in a complete form by Chev. Faà de Bruno. See his valuable *thesaurus* "Théorie des formes binaires," (Turin, 1876,) p. 130, and again p. 310. So much of it as is contained in Salmon's classic work had been previously stated by Cayley in Crelle. J. J. S.

† See Salmon, *Higher Algebra*, 3d ed., page 59, art. 63; Sylvester, "On a Generalization of Taylor's Theorem," *Philosophical Magazine*, Aug., 1877; *Jahrbuch uber die Fortschritte der Mathematik*, vol. 9, p. 171.

a_1, b_1, $c_1 \ldots$, where $a_1 = a$, $b_1 = b + ah$, $c_1 = c + 2bh + ah^2$, $d_1 = d + 3ch + 3bh^2 + ah^3 \ldots$, and if Ω represents the operation $a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \ldots$, then

$$f_1 = f + h\Omega f + \frac{1}{2}h^2\Omega^2 f + \frac{1}{2\,.\,3}h^3\Omega^3 f + \ldots\text{.''}$$

Here, of course, h stands for our x, Ω for our D, a for our $\varkappa$, b for our ι, and so on.

Symbolic demonstrations of these theorems are not difficult to obtain. Thus, let u_y be an arbitrary function of y. Employing the symbol of enlargement E^n, representing an operation such that $\mathrm{E}^n\phi y = \phi(y + n)$, we have $u_1 = \mathrm{E}u_0$, $u_2 = \mathrm{E}^2 u_0$, and so on. Let ψ be any function of u_0, $\mathrm{E}u_0$, $\mathrm{E}^2 u_0 \ldots$, and in this function let us treat the symbol E, wherever it occurs, as itself the subject of operation. Employing a slightly different type, let E represent the operation of enlargement with respect to E, so that $\mathsf{E}^x\phi\mathrm{E} = \phi(\mathrm{E} + x)$, and, similarly, let $\mathsf{D} = \log\mathsf{E} = \frac{d}{d\mathrm{E}}$, representing differentiation with respect to E. Then, since

$$\mathsf{E}^x = 1 + x\mathsf{D} + \frac{1}{2}x^2\mathsf{D}^2 + \frac{1}{2\,.\,3}x^3\mathsf{D}^3 + \ldots,$$

$$\mathsf{E}^x\psi = \psi + x\mathsf{D}\psi + \frac{1}{2}x^2\mathsf{D}^2\psi + \frac{1}{2\,.\,3}x^3\mathsf{D}^3\psi + \ldots\,.$$

It will be seen, on examination, that this is Salmon's theorem. The operation E^x changes E, wherever it occurs in ψ, into $\mathrm{E} + x$, and ψ becomes the same function of u_0, $(\mathrm{E} + x)\,u_0$, $(\mathrm{E} + x)^2\,u_0, \ldots$ that it was before of u_0, $\mathrm{E}u_0$, $\mathrm{E}^2 u_0 \ldots$ That is to say,

u_0 is replaced by u_0,
$u_1 = \mathrm{E}u_0$ " " " $(\mathrm{E} + x)\,u_0 = u_1 + xu_0$,
$u_2 = \mathrm{E}^2 u_0$ " " " $(\mathrm{E} + x)^2\,u_0 = u_2 + 2xu_1 + x^2 u_0$,

and so on. Let $\phi_n(\mathrm{E}^n u_0)$ express ψ as a function of $\mathrm{E}^n u_0$, all the other variables involved in ψ being regarded as constant. Then*

$$\mathsf{D}\psi = \mathsf{D}\phi_1(\mathrm{E}u_0) + \mathsf{D}\phi_2(\mathrm{E}^2 u_0) + \ldots\,.$$

Since

$$\mathsf{D}\phi_n(\mathrm{E}^n u_0) = \phi'_n(\mathrm{E}^n u_0)\,\mathsf{D}\mathrm{E}^n u_0 = \phi'_n(\mathrm{E}^n u_0)\,n\mathrm{E}^{n-1}u_0 = nu_{n-1}\frac{d\psi}{du_n},$$

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*It is unnecessary to discuss the well-known principles by which the legitimacy of these steps is established. Any reader to whom they are not familiar may refer to the "Essay on the Calculus of Enlargement" recently printed in this Journal.

we have, after substitution,

$$\mathsf{D}\psi = u_0 \frac{d\psi}{du_1} + 2u_1 \frac{d\psi}{du_2} + 3u_2 \frac{d\psi}{du_3} + \dots ;$$

or, since ψ may be any function of the variables in question,

$$\mathsf{D} = u_0 \frac{d}{du_1} + 2u_1 \frac{d}{du_2} + 3u_2 \frac{d}{du_3} + \dots .$$

Or, similarly, let $\mathrm{D} = \frac{d}{d0}$, and let

$$v_z = u_0 + u_1 z + \frac{1}{2} u_2 z^2 + \frac{1}{2 \, . \, 3} u_3 z^3 + \dots ,$$

then $u_0 = v_0$, $u_1 = \mathrm{D}v_0$, $u_2 = \mathrm{D}^2 v_0$, and so on, and if we make E such that $\mathsf{E}^x \phi \mathrm{D} = \phi(\mathrm{D} + x)$, and $\mathsf{D} = \log \mathsf{E} = \frac{d}{d\mathrm{D}}$, the demonstration will proceed as before. If, on the other hand,

$$v_z = u_0 + u_1 z + u_2 z^2 + u_3 z^3 + \dots ,$$

we shall, proceeding as in the last case, obtain a theorem equivalent to (1), provided fx is an algebraic function.